using radiation from periodic perturbations as described by Horn *et al.* Further tests are continuing with the fabrication of similar modulators with reduced dimensions for millimeter-wave applications.

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An Expansion of the Terakado Solution with an Application

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***Abstract*—The capacitance of a concentric, symmetrical, rectangular coaxial line in which the outer conductor differs from the inner conductor by a factor of two is expanded to the eleventh power in $\exp[-\pi w/b]$. Here w is the width of the inner conductor and b is the height of the outer conductor. Approximate values obtained from this expansion agree with exact values within 0.06 percent for $w/b > .2$.**

This expansion permits the determination of the limiting value, as $w/b \to \infty$, of the error in an approximation for the characteristic impedance of those rectangular coaxial lines in which the thickness of the inner conduct is half the height of the outer conductor. It is then shown how this information can be used to improve the accuracy with which the characteristic impedance of rectangular coaxial lines may be approximated in the general case.

I. Introduction

Terakado [1] has made the perceptive observation that the transformation

$$z = \frac{1 - \mathrm{cn}(Z,k)}{\mathrm{sn}(Z,k)} \tag{1}$$

which maps the interior of a rectangle, shown in Fig. 1, of width $2K$, and height $2K'$ centered at the origin of the Z-plane onto the unit circle of the z-plane, also maps the L-shaped portion of the rectangle which remains after the lower right-hand cross-hatched quarter of the rectangle has been removed onto a sector of the unit circle. The interior of this sector is mapped by the successive transformations

$$W = z^{2/3} \tag{2}$$

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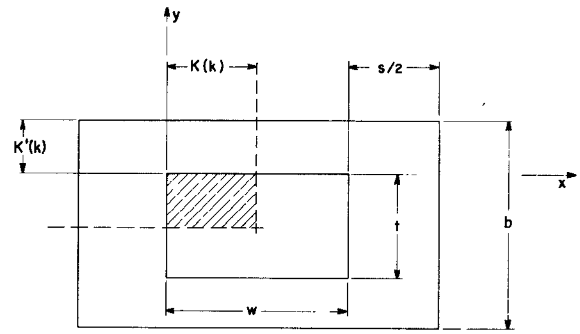

Fig. 1. Z-Plane.

and

$$w = \frac{1}{2}\left(W + \frac{1}{W}\right) \tag{3}$$

on to the lower half of the w-plane. Thus, the transformations (1), (2), and (3) map an L-shaped region of the Z-plane on to the lower half w-plane and permit the exact determination of the capacitance of a class of symmetrical rectangular coaxial transmission lines. The capacitance of these structures is given by

$$C_0 = \frac{4K'(k_0)}{K(k_0)} \tag{4}$$

where

$$k_0^2 = \frac{[1-\cos(\pi/3 - 2\alpha/3)][1-\cos(2\alpha/3)]}{[1+\cos(\pi/3 - 2\alpha/3)][1+\cos(2\alpha/3)]} \tag{5}$$

and

$$\cos(\alpha) = k. \tag{6}$$

In the familiar w, s, t, b notation of Fig. 1, this family of rectangular coaxial lines may be defined by $t/b = 0.5$ and $w/b = s/b$. It is a one parameter family of structures. It depends only on the parameter k determined by the requirement that

$$\frac{K(k)}{K'(k)} = \frac{2w}{b}. \tag{7}$$

It is the immediate object of this paper to present the expansion of the capacitance C_0 of (4) in powers of $\exp(-\pi w/b)$, and to show that this expansion is sufficiently accurate for most purposes. This paper complements papers of Riblet [2]–[4] which present expansions of the capacitances of other well-known rectangular structures directly in terms of their dimensions. Other objectives will be discussed in Section III.

II. The Expansion

It is essential to introduce the nome q' of Jacobi's theory of theta functions. By definition, $q' = \exp(-\pi K/K')$ so that from (7)

$$q' = \exp(-2\pi w/b). \tag{8}$$

For large values of w/b, q' and k' are small so that it is convenient to replace (6) by

$$\sin(\alpha) = k'. \tag{9}$$

This permits an expansion of α in powers of k', which is convergent for small values of k'

$$\alpha = k' + \frac{k'^3}{6} + \frac{3k'^5}{40} + \cdots. \tag{10}$$

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TABLE I
EXACT AND APPROXIMATE C_0
w/b

	1	.15	.2	.3	.4
exact	242	1757	14 354	11.47827	10.469020
approx.	23.6	17.51	14 346	11.47814	10 469018

Moreover, from [5, p. 241]

$$\sqrt{k'} = 2q'^{1/4}\frac{1+q'^2+q'^6+q'^{12}+\cdots}{1+2q'+2q'^4+2q'^9+\cdots}. \tag{11}$$

If p' is defined as $\sqrt{q'}$, $p' = \exp[-\pi w/b]$, and (11) is substituted in (10), a convergent expansion for α in terms of $\exp[-\pi w/b]$ is obtained. This expansion has been carried as far as

$$\alpha = 4p' - \frac{16}{3}p'^3 + \frac{24}{5}p'^5 - \frac{32}{7}p'^7 + \frac{52}{9}p'^9 - \frac{48}{11}p'^{11} + \frac{56}{13}p'^{13} + \cdots. \tag{12}$$

The substitution of (12) in (5) to obtain k_0 is simplified by replacing (5) with

$$k_0'^2 = \frac{3\cos(2\alpha/3)+\sqrt{3}\sin(2\alpha/3)}{\left[1+\frac{1}{2}\cos(2\alpha/3)+\frac{\sqrt{3}}{2}\sin(2\alpha/3)\right][1+\cos(2\alpha/3)]} \tag{13}$$

If $q_0 = \exp(-\pi K'(k_0)/K(k_0))$, from [6, p. 486]

$$q_0 = \frac{1}{2}\frac{1-\sqrt{k_0'}}{1+\sqrt{k_0'}} + 2\left\{\frac{1}{2}\frac{1-\sqrt{k_0'}}{1+\sqrt{k_0'}}\right\}^5 + \cdots \tag{14}$$

while from (4)

$$C_0 = -\frac{4}{\pi}\ln(q_0). \tag{15}$$

The desired expansion then results from the substitution of (12) in (13) to find k_0', followed by the substitution of k_0' in (14) to find q_0 and finally the substitution of q_0 in (15) to obtain C_0.

The substitution of (12) in (13) results in an expansion of k_0' which converges for sufficiently large values of w/b. Moreover, $k_0' \to 1$ as $w/b \to \infty$. The substitution of (13) in (14) also gives an expansion which converges for sufficiently large values of w/b; but $q_0 \to 0$ as $w/b \to \infty$ so that the expansion obtained when (14) is substituted in (15) contains a logarithmic singularity. When this zero factor is removed the remainder of the expansion is convergent.

These substitutions have been carried out in closed form, for reasons to be seen, as far as $\exp(-4\pi w/b)$ and then in decimal form. It has been found that

$$\frac{C_0}{4} = \frac{2\omega}{b} + \frac{1}{\pi}\left\{3\ln(3) + \frac{32}{9}\sqrt{3}\,\omega + \frac{160}{27}\omega^2 + \frac{2432}{729}\sqrt{3}\,\omega^3 + \frac{3808}{729}\omega^4\right\} + 1.61663\omega^5 + 1.54768\omega^6 + 1.39760\omega^7 + 1.38050\omega^8 + 1.40867\omega^9 + 1.41762\omega^{10} + 1.53648\omega^{11} + \cdots \tag{16}$$

where $\omega = \exp(-\pi w/b)$.

Table I, in which the values in the upper row are exact and the values in the lower row are obtained from (16), indicates that the error in (16) is rapidly decreasing function of w/b. The error which is 2.5 percent for $w/b = 0.1$ reduces to 0.34 percent when $w/b = 0.15$ while seven place agreement is found when $w/b = 0.4$.

III. THE APPLICATION

It is interesting and useful to evaluate the difference between this solution and the approximations introduced by Barrett [7] and Cohn [8] and Riblet [9]. If ΔC_{f_0} denotes the difference between the approximate fringing capacitance C_{f_0}' and the exact fringing capacitance C_{f_0}, then the total capacitance C_0 of a rectangular coaxial line is given by

$$C_0 = 4\{w/(b-t) + C_{f_0}' - \Delta C_{f_0}\}. \tag{17}$$

Riblet [3, eq. 12] has given an expansion of C_{f_0}' in the desired form. When $t = b/2$, $\beta = 2$; and if $\omega = \exp(-\pi s/b) = \exp(-\pi w/b)$

$$C_{f_0}' = \frac{1}{\pi}\left\{3\ln(3) + \frac{32}{9}\sqrt{3}\,\omega + \frac{160}{729}\omega^2 + \frac{2432}{729}\sqrt{3}\,\omega^3 + \frac{3872}{729}\omega^4\right\} + 1.70267\omega^5 + 1.69671\omega^6 + 1.6398/\omega^7 + \cdots. \tag{18}$$

Then, if (17) is solved for ΔC_{f_0}, when C_0 and C_{f_0}' are replaced by their expansions in terms of ω

$$\Delta C_{f_0} = \frac{1}{\pi}\frac{64}{729}\omega^4 + 0.08604\omega^5 + 0.14903\omega^6 + 0.24221\omega^7 + \cdots. \tag{19}$$

If $\Delta' C_{f_0}$ is defined by the equation

$$C_0 = C_s + 4(C_{f_0}' - C_f' - \Delta' C_{f_0}) \tag{20}$$

where C_s is the capacitance of the slab-line structure for the same values of t/b and $w/(b-t)$, and $C_{f'}$ is the limiting value of C_{f_0}' as $s/b \to \infty$, then $\Delta' C_{f_0}$ can also be expanded in this form since Riblet has given the required expansion for C_s in [2, eq. 20]. In the present case, $b = 2t$ and $\alpha = 0.5$. Then

$$\frac{C_s}{4} = \frac{2\omega}{b} + \frac{1}{\pi}3\ln(3) - \frac{64}{729}\omega^4 - 0.024467\omega^8 + \cdots. \tag{21}$$

When (20) is solved for $\Delta' C_{f_0}$ and the expansions for C_0, C_s and C_{f_0}' are substituted in the result

$$\Delta' C_{f_0} = 0.08604\omega^5 + 0.14903\omega^6 + 0.24221\omega^7 + \cdots \tag{22}$$

since, by definition, C_f' is the limit of C_{f_0}' as $\omega \to 0$.

It is clear that $\Delta' C_{f_0} < \Delta C_{f_0}$ in this case, as expected, but of greater interest is the fact that (19) and (22) indicate that the approximation of the total capacitance of rectangular coaxial line in terms of parallel plate, slab-line, and fringing capacitance has a fundamental analytical basis. Moreover, since the expansion for C_0 is known to converge, the fact that the expansions for C_{f_0}' and C_s when properly combined have the same first five terms strongly suggests that they are also convergent.

The information provided by the Terakado solution permits one to plot curves of $\Delta' C_{f_0}\exp(2\pi w/(b-t))$ for the case, $t/b = 0.5$, which allow a significant improvement in the accuracy of the improved approximation for the characteristic impedance of rectangular coaxial line recently presented by Riblet [9]. Table II shows a number of exact values on these curves. The values in the first column when the inner conductor is a zero thickness, vertical strip may be obtained from Oberhettinger and Magnus [10, pp. 62–65] while their limiting values as $s/b \to \infty$ can be determined from Riblet [4, pp. 662–663]. The values in the box bordered by double lines were obtained from the exact case discussed by Bergmann [11, pp. 319–331]. The values in the right-hand region are exact values obtained from the Terakado solution which were

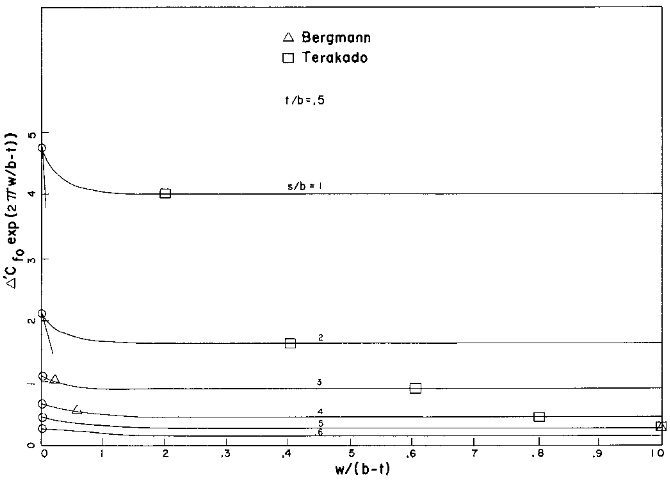

Fig. 2. Curves of $\Delta' C_{f_0} \exp(2\pi w/(b-t))$.

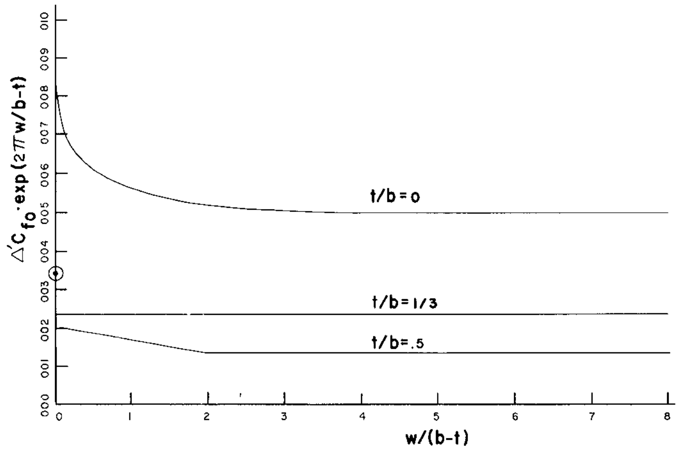

Fig. 3. Curves of $\Delta' C_{f_0} \exp(2\pi w/(b-t))$ for $s/b = 4/3$.

TABLE II
EXACT VALUES OF $\Delta' C_{f_0} \cdot \exp(2\pi w/(b-t))$ FOR $t/b = 0\ 5$

$w/(b-t)$

s/b	0	603E-7	315E-3	727E-2	504E-1	1	2	4	6	8	1	>1
1	4798539	4798529					3988					
2	2134		2124					1646				
3	1116			1057					07972			
4	06473				05368					04450		
5	04031					02712					02712	
6	02636											01752
7	01782											01175
8	01233											008089
9	008663											005666
10	006155											004017
>1	$1325e^{-\pi s/b} + 2159e^{-2\pi s/b} + 3127e^{-3\pi s/b}$						$08604e^{-\pi s/b} + 1490e^{-2\pi s/b} + 2423e^{-3\pi s/b}$					

obtained by direct numerical computation. The limiting values shown below are given by (22) after multiplication by $\exp(2\pi w/(b-t)) = \exp(4\pi s/b)$.

Curves plotted on the basis of this data are shown in Fig. 2 where the points denoted by squares are provided by the Terakado solution while those shown by the triangles depend on Bergmann. For $s/b = 0.1$ and 0.2, these values were used to determine the slope of the curves at $w/(b-t) = 0$. They show that the majority of the change in the curves occurs for $w/(b-t) < 0.2$. Thus, the Terakado results determine the curves over the greater part of their extent. For $s/b = 0.5$, the Bergmann and Terakado results coincide.

These curves or other curves of the same family which can be plotted by the same method always provide a more accurate approximation of $\Delta' C_{f_0}$ and thus of C_0 than was given in [9]. In the first place, the curves of Fig. 2 confirm the argument made in [9] that the $\Delta' C_{f_0} \exp[2\pi w/(b-t)]$ are decreasing functions of s/b and t/b. In fact, a comparison of these curves with those of [9, fig. 2] shows that, for a given value of s/b, the curve for $t/b = 0.5$ has a value about one half the value of the corresponding curve for $t/b = 0$. Thus, for $t/b > 0.5$, the use of these curves as an upper limit, instead of those for $t/b = 0$ as was done in [9], will reduce the error in the approximation by a factor of two.

It is noteworthy that all of these curves, both for $t/b = 0.5$ and $t/b = 0$ have their maximum value when $w/(b-t) = 0$. It is clear that this is not a characteristic which is peculiar to the case when $t/b = 0$ and may be reasonably expected to hold for all t/b since it undoubtedly results from the rapid change in the field distributions as the width of the inner conductor approaches zero.

How this information, together with the curves of Fig. 2, can be used to improve the approximation when $0 < t/b < 0.5$ will be illustrated by considering the example where $w/(b-t) = 1$, $s/b = 4/3$ and $t/b = 1/3$. This was the case selected by Cruzan and Garver [12, p. 495] to illustrate the use of their graphs to determine the characteristic impedance of rectangular coaxial line. In Fig. 3, three curves of $\Delta' C_{f_0} \exp(2\pi w/(b-t))$ are plotted for the case when $s/b = 4/3$. The upper curve was plotted using exact values for the case $t/b = 0$, while the lower curve was drawn between the end points determined from the two expansions on the bottom line of Table II for $s/b = 4/3$. Also plotted is the one known point on the curve for which $s/b = 4/3$ and $t/b = 1/3$, namely the point at $w/(b-t) = 0$, where it's value is 0.003374. It has been argued that the quantity $\Delta' C_{f_0} \exp[2\pi w/(b-t)]$ for the case $s/b = 4/3$ and $t/b = 1/3$ has its maximum value at $w/(b-t) = 0$. Moreover, since these are decreasing functions of t/b, this curve must lie above the corresponding curve for $t/b = 0.5$. In fact, it must lie entirely above the value 001340 given by the terms in the lower right-hand corner of Table II. It can certainly be approximated by the horizontal line drawn halfway between its maximum value at $w/(b-t) = 0$ and the minimum value of the curve for $t/b = 0.5$. Thus

$$\Delta' C_{f_0} \exp(2\pi w/(b-t)) = 0.002357 \pm 0.001017 \tag{23}$$

for $s/b = 4/3$, $t/b = 1/3$, and all values of $w/(b-t)$. For the case when $w/(b-t) = 1$, $\Delta' C_{f_0} = (4.402 \pm 1.900)\ 10^{-6}$. Then[1]

$$C_0 = 7.454038618 \pm 0.000007597. \tag{24}$$

Of course it is not surprising that this approximation should be very accurate for a value of $w/(b-t)$ as large as 1. Consider the case where $w/(b-t) = 0.1$. Then $\Delta' C_{f_0} = 0.001257 \pm 0.000543$, and

$$C_0 = 3.76456 \pm 0.00217 \tag{25}$$

so that the characteristic impedance can be determined with an accuracy better than 0.06 percent even for this small value of $w/(b-t)$.

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Radiation Resistance in Radial Transducer

E. SAWADO

I. Introduction

The purpose of the present paper is to give an intuitive explanation for the characteristics of radial wave and to demonstrate that this mode has no cutoff below the critical frequency $\omega = \gamma(BH)^{1/2}$, where ω is the angular frequency, $\gamma = 1.76 \times 10^7$ ((oe sec)$^{-1}$ in CGS unit), $B = \mu_0(H+M)$ is the magnetic flux density, H the magnetic field, M the saturation magnetization. Ganguly and Webb, and others [1]–[3] presented an initial theory and experiments for magnetostatic surface wave transducers. They obtained some of the useful results for resistance of a microstrip due to radiation. Previous investigations have calculated dispersion characteristics [4] and characteristic impedance of composite microstrip slab structure [5]. These investigations conclude that microstrip excitation of magnetostatic surface wave has proven particularly convenient, because of strong coupling from electromagnetic waves to magnetostatic waves. It is easy to see that the lowest operating frequency of the Ganguly type delay line is γH. Below this cutoff, no modes can exist. In view of the above, investigation of radial wave type delay line should produce useful developments in low frequency microwave (0.5 to 1.0 GHz) applications.

The system analyzed in this report is shown in Fig. 1. A transducer in the form of a fine wire is excited with an RF current which generate radial volume waves within the structure. The dc magnetic field is directed along the z axis and also the fine wire is situated parallel to the z axis. This mode propagates perpendicular to the magnetic biasing fields, guided by two parallel surfaces, and its energy is distributed within the medium.

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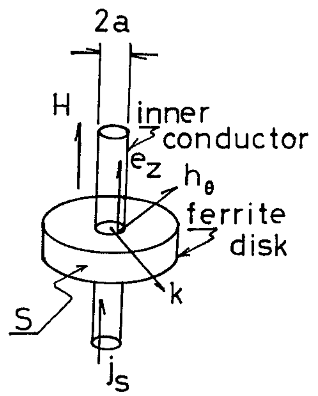

Fig. 1. Radial waveguide and an infinitely long wire as exciting antenna.

II. Basic Theory

Coupled differential equations for the z component of electric and magnetic fields are [6], [7].

$$\left(\nabla_\perp^2 + \frac{\partial^2}{\partial z^2} + \omega^2 \epsilon \mu_0 \mu_\perp\right) e_z + \omega \mu_0 \kappa / \mu \frac{\partial}{\partial z} h_z = 0 \quad (1)$$

$$\left(\nabla_\perp^2 + (1/\mu) \frac{\partial^2}{\partial z^2} + \omega^2 \epsilon \mu_0\right) h_z + \omega \epsilon \kappa / \mu \frac{\partial}{\partial z} e_z = 0 \quad (2)$$

where μ_0 and ϵ are the vacuum permeability and dielectric constant, respectively, $\nabla_\perp^2$ is a differential operator $= \partial^2/\partial x^2 + \partial^2/\partial y^2$, and $\mu_\perp$ is the effective permeability given by

$$\mu_\perp = (\mu^2 - \kappa^2)/\mu \quad (3)$$

where μ and κ are a diagonal and a nondiagonal component of the relative permeability tensor. These equations become two independent differential equations, when the fields are independent of $z(\partial/\partial z = 0)$. The assumption that $(\partial/\partial z = 0)$ is true only under two conditions: 1) when there are no energy leaks into the free space, and 2) when the thickness of the ferrite sheet is small compared with wavelength λ. We now express them in cylindrical coordinates

$$\left(\frac{1}{\sigma}\frac{\partial}{\partial \sigma}\left(\sigma \frac{\partial}{\partial \sigma}\right) + \omega^2 \epsilon \mu_0 \mu_\perp\right) e_z = 0 \quad (4)$$

$$\left(\frac{1}{\sigma}\frac{\partial}{\partial \sigma}\left(\sigma \frac{\partial}{\partial \sigma}\right) + \omega^2 \epsilon \mu_0\right) h_z = 0. \quad (5)$$

From (4), we have a solution for a radial argument in terms of the Bessel function of order 0 and the Hankel function of the second kinds of order 0

$$e_z = a_0 J_0(k\sigma) + b_0 H_0^{(2)}(k\sigma). \quad (6)$$

Substituting (6) into (4), we have

$$k^2 = \omega^2 \epsilon \mu_0 (\mu^2 - \kappa^2)/\mu. \quad (7)$$

This relation gives a dispersion of this system. The boundary condition on the z component of electric field requires that

$$e_z = 0 \qquad \text{when } \sigma = a. \quad (8)$$

The z component of the electric field satisfying the boundary